The Bounded Trajectories Conjecture for Syracuse Maps

Syracuse Maps as Power-Bounded and Asymptotically Mean Stationary Operators, and Considerations on the Densities of Relevant Sets

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Honors Thesis
Mathematics Department
University of North Carolina Chapel Hill
2024

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Abstract

The Collatz Conjecture has haunted the mathematical scene for nearly 100 years. In this thesis, we introduce part of the conjecture, and its generalization to a larger class of maps. Using some theory from dynamical systems, we pose several equivalent conditions to conjecture through the construction of various measures. Next, we investigate the set L of Terras, providing a rate of convergence to his density argument.

Acknowledgements

I would like to thank David Niu, who first invited me to talk to meet Professor Assani and work with the Collatz Map. His enthusiasm started my work in this project.

I would like to thank Dr. Idris Assani, my advisor of three years whose extraordinary penchant for teaching and deep love of research made this work possible. Further, I would like to thank Anand Hande, my research partner for two of these three years that worked with me on sections 3.1, 3.2, and 3.3. His vigor kept this project moving even despite initial failures.

I would like to thank my committee members, Dr. Mark Williams and Dr. Richard Rimanyi, both for their efforts to review this work and their contributions to my development as a mathematician in the classroom.

Finally, I would like to acknowledge the support of the Knoble Family Excellence Fund for Honors, provided by Honors Carolina.

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1 The Collatz Conjecture and Some Notable Contributions

The Collatz Conjecture is named after Lothar Collatz, who introduced the problem in 1937, and may also be referred to as Kakutani's problem or Ulam's Conjecture. It focuses on a pointwise-defined map called the Collatz map $T : \mathbb{N} \to \mathbb{N}$, given by

$$T(x) = \begin{cases} \frac{3x+1}{2} & x \text{ odd} \\ \frac{x}{2} & x \text{ even} \end{cases} \tag{1}$$

Note, for example, that $T(1) = 2$ and $T(2) = 1$. We call this a cycle of the Collatz map. More precisely, a cycle is a finite collection of elements $\{c_1, ..., c_n\}$ so that $T(c_i) = c_{i+1}$ for $i < n$ and $T(c_n) = c_1$. The first part of the Collatz Conjecture poses that $\{1, 2\}$ is the only cycle of the Collatz map.

Of course, some numbers are not in cycles, like 3. We have

$$\begin{aligned} T(3) &= 5 \\ T^2(3) &= 8 \\ T^3(3) &= 4 \\ T^4(3) &= 2 \end{aligned}$$

The trajectory of 3 then enters the cycle $\{1, 2\}$. In general, the value $x \in \mathbb{N}$ is said to have bounded trajectories if $T^k(x)$ is part of some cycle for some $k \in \mathbb{N}$. The second part of the Collatz Conjecture poses that every natural number has a bounded trajectory under the Collatz map.

It is known through supercomputer computation that all numbers up to 1642×2^{60} return to the cycle $\{1, 2\}$ [4]. Furthermore, there has been a litany of work on the topic, investigating paths tying the problem to stochastic processes, p-adic numbers, Ergodic Theory, high-dimensional geometry, and more [12].

In lieu of direct proofs, many authors have turned to showing desirable properties on sets of density 1. Recall that a set $A \subset \mathbb{N}$ is said to have natural density d if

$$\lim_{n\to\infty} \frac{card(A \cap [0, n])}{n} = d \tag{2}$$

We consider three main results here. R.Terras [20] first showed that the set $L = \{x \in \mathbb{N} \mid T^k(x) < x \text{ for some } k \in \mathbb{N}\}$ has natural density 1 through an argument using random variables. I.Korec [8] extended the set L to a set $M_C = \{x \in \mathbb{N} \mid T^k(x) < x^c \text{ for some } k \in \mathbb{N}\}$, and showed that for $c > \log_4(3)$, that M_C has natural density 1 through a combinatorial limit argument. Further analysis into these densities via statistical properties and probability distributions have been carried out by authors such as Lagarias [11], Kontovorich, and Sinai [7,18]. Finally, T.Tao [19] showed that for any $f : \mathbb{N} \to \mathbb{R}$ so $\lim_{n\to\infty} f(n) = \infty$, that the set $\{x \in \mathbb{N} \mid T^k(x) < f(x) \text{ for some } k \in \mathbb{N}\}$ has logarithmic density 1, using an argument employing random variables and representation theory.

The Collatz Conjecture has also been extended beyond the Collatz map. The same problem has been posed for a broader class of maps we will call Syracuse maps. We rely on the definition for this class of maps from Matthews and Watts [15] given by

$$V(x) = \begin{cases} \frac{m_i x + r_i}{d} & x \equiv i \mod d \end{cases}$$

where $r_i \equiv -im_i \mod d$ and $m_0m_1...m_{d-1}$ is relatively prime to d. We will not need this relatively prime assumption, but it allows some useful number-theoretic properties in other cases. These maps may have more or different cycles than $\{1, 2\}$, so in the case of these maps, authors tend to focus only on showing they have bounded trajectories.

This paper will focus on approaching the second part of the Collatz Conjecture from an Ergodic viewpoint. Section 2 will consider this viewpoint explicitly, beginning with a theorem of I.Assani [1]. Theorems 4 and 5 will extend those constructions to a broader class of point-maps, following the steps of the author, I.Assani, and A.Hande in [3]. We will then present some combinatorial considerations on the Collatz map and class of measures connected to them, providing the motivation for two separate characterizations of the Collatz map, Propositions 8 and 10, as presented by the author in [2].

We will then shift to considering steps toward approaching the Collatz conjecture from this viewpoint by tracing the behaviors of the inverse Collatz map. In particular, Section 3 showcases the Triangle conjecture. Propositions 15 and 16 will reduce the conjecture to the consideration of a single case (as presented in [2]). We also consider the "shape" of the preimages of a point, in particular the relative sizes of preimage levels, which is being first presented in this thesis. Proposition 18 and Lemma 22 will sandwich the inverse images between two sequences of asymptotically geometric growth, then consider this in the associated triangle.

Finally, we will consider a connection of the triangle conjecture to density results. By strengthening the result of R.Terras, Theorem 24 will give sharper bounds on the density of values with bounded stopping times. Corollary 28 then connects this directly back to a step toward the Triangle Conjecture. These results both appear in the author's work [2], and tie into a deeper combinatorial structure whose considerations are currently in preparation.

2 Ergodic Viewpoint

This is certainly not the first time the Collatz map has been viewed under the lense of Ergodic Theory. Indeed, the map has been viewed as a discrete dynamical system by Wirsching [21] and others such as Matthews [13–15] and Lagarias [6, 10, 11] have developed theory around the Collatz map in Ergodic theory. The underpinnings of Ergodic theory have influenced a number of other authors as well. Here, we view the Collatz problem specifically in the environment of power-bounded operators, and use that to recharacterize the conjecture for a broad class of maps.

2.1 Introduction: Decomposition of the Natural Numbers and its Consequences

We begin by noting some of the foundational work of I.Assani to this topic in [1]. These cases are implied by results in the next section, so we elide their proofs and focus on intuition here. First, we need some definitions.

Definition 1. *Let μ be a measure defined on a σ-algebra $\mathcal{A}$ on the set X. Let $T : X \to X$ be a map. Then, the quartuple $(X, \mathcal{A}, T, \mu)$ is said to be a dynamical system. This system is said to be nonsingular or null-preserving if $\mu(T^{-1}(A)) = 0$ whenever $\mu(A) = 0$.*

The focus of a dynamical system is on the repeated behavior of the map as it connects to the measure. Most results posed in dynamical systems thus act up to sets of measure 0, but that will not constrain anything for our purposes. Nonsingularity simply acts to preserve these null properties, where the inverse map connects this preservation to integration (where one might assume we want $T(A)$ to be null in other contexts).

Definition 2. *: Let $(X, \mathcal{A}, T, \mu)$ be a non-singular dynamical system. It is said to be power bounded in $L^1(\mu)$ (or often simply just power bounded) if there exists some $M \in \mathbb{R}_+$ such that for all sets $A \in \mathcal{A}$ and natural numbers n, $\mu(T^{-n}(A)) \leq M\mu(A)$.*

A dynamical system being power bounded roughly means that the preimages don't pick up too much mass as they continue. For now, we will consider the dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, T, \mu)$ for some measure μ and where T is the Collatz map.

With this dynamical system, we may partition the natural numbers in a sensible way. We consider all of the cycles of the Collatz map, and denote the set of all points in the cycles as C. Then, we consider all the points in the preimages of the cycles under repeated iterations of the Collatz map, or all points that map into a cycle eventually, and call this set D_1. Everything leftover from that, we call D_2. Under this description, it is clear that the set D_2 contains everything never mapping into a cycle, or all values with unbounded trajectories. Therefore, the second part of the Collatz Conjecture translates to showing D_2 empty. Using the properties of a power-bounded system, this leads to the characterization

Theorem 3. *The set D_2 is empty if and only if there exists some finite measure μ equivalent to the counting measure such that $(\mathbb{N}, 2^{\mathbb{N}}T, \mu)$ is a power-bounded dynamical system.*

Recall that two measures are equivalent if they are absolutely continuous with respect to each other. For this theorem, being equivalent to the counting measure is to say that $\mu(n) > 0$ for all $n \in \mathbb{N}$. In context, such a measure may be viewed as merely a function $f : \mathbb{N} \to \mathbb{R}^+$ so $\sum_n f(n) < \infty$.

This theorem then says the second part of the Collatz conjecture may be proved by constructing such a measure. This puts a lighter restraint on solving that part of the conjecture than previous works, which sought a to make $(\mathbb{N}, \mathcal{A}, T, \mu)$ a measure-preserving system. Let us both generalize and formalize this argument.

2.2 Extension to Syracuse Maps

The decomposition above is motivated by the Hopf Decomposition, though it is not strictly necessary to use it. Colloquially, the Hopf decomposition provides that, in a non-singular system, the ambient space may be separated into sets conservative and dissipative with respect to the map (for a full proof and rigorous statement, see [9]). These properties explain the notation of C and $D = D_1 \cup D_2$ used above. In the Collatz map, the conservative part is precisely the set of cycles. We now sharpen the statement.

Theorem 4. *Consider a non-singular dynamical system $(\mathbb{N}, 2^{\mathbb{N}}, V, \nu)$. There exists a partition of $\mathbb{N}$ into sets C, D_1, D_2 such that:*

1. *The restriction of V to C is conservative (for any set σ of positive measure, $\nu(\sigma \cap T^{-n}(\sigma)) > 0$ for some n). The set C is V-absorbing, and is the at-most-countable union of cycles C_i.*
2. *The set D_1 is equal to $\bigcup_{k=1}^{\infty} V^{-k}(C) \backslash C$.*
3. *The set D_2 is the complement of $C \cup D_1$ in $\mathbb{N}$, $V^{-1}(D_2) = D_2$.*
4. *Any and all unbounded trajectories of V lie in D_2*

Proof. : The Hopf Decomposition provides a partition of $\mathbb{N}$ into sets C and D, so that the restriction of V to C is conservative and C is V-absorbing. Since $\mathbb{N}$ is countable, C also is, so it can contain at most countably many cycles. This proves (1). Let $D_1 = \bigcup_{k=1}^{\infty} V^{-k}(C) \backslash C$. Let $D_2 = \mathbb{N} \backslash [C \cup D_1]$. Then, $V^{-1}(C \cup D_1) = V^{-1}(C) \cup \bigcup_{k=1}^{\infty} V^{-k-1}(C) \backslash C = C \cup D_1$. Therefore, $V^{-1}(D_2) = D_2$, and (2) and (3) are proven. Finally, given any $x \in \mathbb{N}$, a bounded trajectory means that there exists n so $V^n(x) \in C$, so either $x \in C$ or $x \in D_1$. Given an unbounded trajectory of the point y, the set $\{y\}$ is wandering, and $V^m(y) \notin C$ for any natural number m. Thus, $y \in D_2$ and (4) is proven. □

With the more general decomposition, we can also prove the more general case of the characterization of unbounded trajectories.

Theorem 5. *Let $(\mathbb{N}, 2^{\mathbb{N}}, V, \mu)$ be a dynamical system where μ is a finite measure equivalent to the counting measure. Then, it is power-bounded in $L^1(\mu)$ if and only if there exists at least one cycle of the map V and for any $x \in \mathbb{N}$, there exists a non-negative integer k such that $V^k(x)$ is in some cycle of the map V (or $C \neq \emptyset$ and $D_2 = \emptyset$).*

Proof. $\Rightarrow$) Consider the power-bounded system $(\mathbb{N}, 2^{\mathbb{N}}, V, \mu)$. By contradiction, let there exist some x so $V^k(x)$ is never in a cycle. Then, for all $k, l > 0$, $k \neq l$, $V^k(x) \neq V^l(x)$. Let $\mu(x) = \delta > 0$. Since the measure is finite, $\mu(\bigcup_{i=0}^{\infty} V^i(x))$ is also finite and is $\sum_{i=1}^{\infty} \mu(V^i(x)) = \epsilon > 0$, implying that $\mu(V^k(x)) \to 0$ as $k \to \infty$. Given any $M \in \mathbb{R}_+$, we may take some large N such that $\mu(V^N(x)) < \delta/M$. Hence, $V^{-N}(V^N(x)) > \delta > M\mu(V^N(x))$, contradicting that the system is power bounded.

$\Leftarrow$) Let $C \neq \emptyset$ and $D_2 = \emptyset$. Since the space is countable, there are at most countably many cycles, and every point is in a cycle or the pre-image of some cycle. Let the cycles be the sets $C_1, C_2, ...$. First consider C_1. We generate a measure in $C_1 \cup \bigcup_{k=1}^{\infty} V^{-k}(C_1)$. The cycle must be of finite length, N. Define a function μ_1, which will have a measure value at each specified point, and set the measure of any $A \in 2^{\mathbb{N}}$ to be the sum of the measures of the points in A. Let $\mu_1(c) = \frac{1}{2N}$ for any $c \in C_1$, so that $\mu_1(C_1) = \frac{1}{2}$. Next, there are at most countably many points in $V^{-1}(C_1)\backslash C_1$. The infinite case implies the finite case, so we consider this case. Enumerate these points as $\{c_1, c_2, c_3 ...\}$. Set $\mu_1(c_i) = 2^{-i-3}$, so that $\mu_1(V^{-1}(C_1)\backslash C_1) \leq 2^{-2}$.

We iterate the same process on the elements of $V^{-2}(C_1)\backslash C_1$. Consider $V^{-1}(c_i)$, which does not contain c_i nor any other point with a defined measure since such a case would generate a cycle. It again may be enumerated as $\{c_{i,1}, c_{i,2}, c_{i,3} ...\}$, since the set is at most countably infinite. Set $\mu_1(c_{i,j}) = 2^{(-i-3)+(-j-2)}$ so that $\mu_1(V^{-1}(c_i)) \leq 2^{-i-4}$ and so $\mu_1(V^{-1}(V^{-1}(C_1)\backslash C_1)) = 2^{-3}$. Repeat inductively over the pre-images generated by this set, and set all points not in $\bigcup_{i=0}^{\infty} V^{-i}(C_1)$ to have measure zero. This generates μ_1.

The process above gives that $\mu_1(\bigcup_{i=0}^{\infty} V^{-i}(C_1)) = \mu_1(C_1) + \mu_1(V^{-1}(C_1)\backslash C_1) + \mu_1(V^{-1}(V^{-1}(C_1)\backslash C_1)) + ... \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 1$. Repeat for each of the C_i to generate μ_i, and let $\mu = \sum_{i=1}^{\infty} 2^{-i-1}\mu_i$, so $\mu(\mathbb{N}) \leq 1$ and μ is a finite measure. Further, every point in $\mathbb{N}$ is in one of the $\bigcup_{i=0}^{\infty} V^{-i}(C_j)$, so every point has nonzero measure under μ. Next, we need to show that the measure allows our map V to be power bounded. By construction, $\mu_i(V^{-1}(A)) \leq \mu_i(A)$ for any set A such that $A \cap C_i = \emptyset$. For a set B intersecting the cycle C_i, separating $B = (B \cap C_i^C) \cup (B \cap C_i)$ gives $\mu_i(V^{-n}(B)) \leq 2^{-n}\mu_i(B \cap C_i^C) + 2\mu_i(B \cap C_i) \leq 2\mu_i(B)$ for all $n \in \mathbb{N}$. Therefore, $\mu(V^{-n}(A)) = \sum_{i=1}^{\infty} 2^{-i-1}\mu_i(V^{-n}(A)) \leq \sum_{i=1}^{\infty} 2^{-i}\mu_i(A) = 2\mu(A)$. The measure μ is finite and power-bounded in $\mathcal{L}^1$, as well as equivalent to the counting measure, as desired.

□

Remark: This proof also immediately shows how we may produce a finite measure so T is measure-preserving using the cycles, and that any invariant measure must be σ-finite if D_2 is nonempty and μ has support intersecting D_2. Indeed, one may also construct an everywhere-nonzero invariant measure relatively easily through a similar argument.

This result show the broad applicability of of this power-bounded system connection, but that generality also shows the difficulty in the case of Syracuse maps, as we may generate maps such as

$$W(x) = \begin{cases} 1 & x = 2 \\ 2 & x = 1 \\ 2x & x > 2 \end{cases} \tag{3}$$

which satisfy the assumptions on the map readily.

2.3 The Chain Structure

The previous theorem leads, of course, to trying to construct such a measure. Because of the generality of the Syracuse version, one would hope to rely on some combinatorial properties of the

Collatz map. In the case of a Syracuse map

$$V(x) = \begin{cases} \frac{m_i x + r_i}{d} & x \equiv i \mod d \end{cases}$$

we may have at most d preimages of any given point. Given a specific case, we may break down the number and forms of each preimage by the class of the image modulo d. We do so in the specific case of the Collatz map.

Definition 6. *A value $n \in \mathbb{N}_i$ is said to be in $\mathbb{N}_i$ if $n \equiv i$ (mod 3).*

Computational checks show directly that the pre-image of a node in $\mathbb{N}_0$ is again in $\mathbb{N}_0$, the pre-image of a node in $\mathbb{N}_1$ is in $\mathbb{N}_2$, and the pre-image of a node in $\mathbb{N}_2$ includes one node either in $\mathbb{N}_1$ or $\mathbb{N}_0$ and another in $\mathbb{N}_1$.

Indeed, we have

$$T^{-1}(x) = \begin{cases} \{2x\} & x \in \mathbb{N}_0 \cup \mathbb{N}_1 \\ \{2x, \frac{2x-1}{3}\} & x \in \mathbb{N}_2 \end{cases}$$

The latter behavior is the most interesting, and may be characterized in a more lucid way, using a simple lemma.

Lemma 7. *Any node in $\mathbb{N}_2$ may be written as $3^a 2^b h - 1$ where $a > 0$, $b \geq 0$, and $h \geq 1$ is not a multiple of 2 or 3.*

Proof. For arbitrary $n \in \mathbb{N}_2$, it may be written as $3p + 2$ be definition, and thus as $3(p + 1) - 1$. Using the Fundamental Theorem of Arithmetic, $p + 1$ has the desired representation $3^{a-1}2^b h$. □

Under this representation, $T^{-1}(3^a 2^b h - 1) = \{3^{a-1}2^{b+1}h - 1, 2(3^a 2^b h - 1)\}$. In particular, we may characterize a family of nodes by either the image or part of the pre-image, where $T(2^a h - 1) = 3(2^{a-1})h - 1$, $T(3(2^{a-1})h - 1) = 3^2 2^{a-2}h - 1$, and this repeats until $T^a(2^a h - 1) = 3^a h - 1$. This repeating pattern may be referred to as a family.

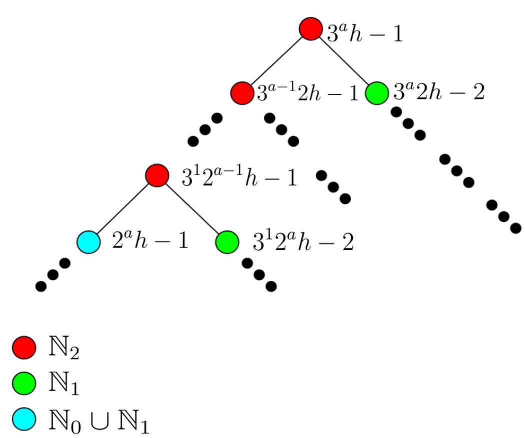


Figure 1: Inverse Image Tree generated by $3^a h - 1$ up to the a^{th} level

In this final case, the node is even and its image is a $\mathbb{N}_1$ node. The pre-image of the $\mathbb{N}_1$ node is again a $\mathbb{N}_2$ node, starting the process again. Starting from some $\mathbb{N}_2$ node $3^a h - 1$, looking at only the preimages shown on the left-most branch in the figure brought enough interest to be coined a "chain", though the use of this term has been extended to refer to any $\{a_z\}_{z\in\mathbb{Z}}$ so $T(a_k) = a_{k+1}$. We also refer to the chain-tree generated by a value $a \in \mathbb{N}$ to be $\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} T^{-j}(T^i(a))$.

This family structure leads to a class of measures which resemble the desired form within these families, but fails to be power-bounded in these connections between the families. For $\alpha, \beta > 1$, this measure takes the form

$$\mu(x) = \begin{cases} \alpha^{-a}\beta^{-b} & x = 3^a 2^b h - 1 \text{ for } h \text{ odd and not a multiple of } 3 \\ \frac{\mu(T(x))}{\beta} & \text{otherwise} \end{cases}$$

This measure relies on the fact that at the preimage tree-structure is determined in chunks dictated by these $3^a h - 1$ nodes, for h odd and not a multiple of 3. The failure of the measure occurs because, between families, we may transfer between a large a in $3^a h - 1$ and a large h. This is revealed immediately by noting that $2^{a+1}h \equiv 4 \mod 9$ for fixed h has solutions for arbitrarily large a, which gives a connection $T^{-2}(2^a h - 1) = \{3h_2 - 1\}$ for h_2 odd and not a multiple of 3. Whether there is a way to rectify this measure has remained open.

The study of the triangular structure motivated by this near-miss inspired the development of this chain theory, leading to some results extending the main theorem of this section presented next. It also inspired a deeper conjecture on the structure of the Collatz map, presented in the next section.

2.4 Extension to Less Restrictive Measures

The above work focused on a posing the Collatz Conjecture in the context of a well-known type of dynamical system, and it relies on a simple trick to do so. It is immediate to ask how the requirements of the above structure may be weakened. In this section, we look at two cases on the Collatz map which place limit-based restrictions on the measure instead of finite restrictions. Both cases may be proved algebraically through use of the chain structure presented above, as well as proved Ergodically. We present the chain-based proof of both and an Ergodic proof of the second.

Proposition 8. *Let there exist a finite measure μ on $\mathbb{N}$ such that $\lim_{n\to\infty} \mu(T^{-n}(A))$ exists for all $A \subset \mathbb{N}$. Then, D_2 must have measure 0.*

Proof. By contradiction, let $a \in D_2$ have $\mu(a) > 0$. Let $E = \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{\infty} T^{-j}(T^i(a))$ be a chain-tree in D_2. We may then pick a chain $H = \{a_z\}_{z\in\mathbb{Z}}$ in E to be a set such that $T(a_z) = a_{z+1}$ and $a_0 = a$. We focus on the measure on H. First, we construct a set $\mathcal{A}_n$. Define $A_n = \{a_z\}_{z\in n\mathbb{Z}}$ to be every n-th node in the chain. We pick $\mathcal{A}_n = \bigcup_{i=1}^{\infty} T^{-in}(A_n)$. For each $\mathcal{A}_n$, the n different sets $\mathcal{A}_n, T^{-1}(\mathcal{A}_n), T^{-2}(\mathcal{A}_n), ...T^{-(n-1)}(\mathcal{A}_n)$ are disjoint, and $T^{-n}(\mathcal{A}_n) = \mathcal{A}_n$. These sets repeat as we take preimages. Therefore, if any two of these sets have different measures under μ, say $\mathcal{A}_n$ and $T^{-m}(\mathcal{A}_n)$, then the sub-sequences of $T^{-k}(\mathcal{A}_n)$ corresponding to these generated by the n^{th} and $n + m^{th}$ indices converge to different values and the proof is complete.

Next, assume that $\mu(\mathcal{A}_n) = \mu(T^{-1}(\mathcal{A}_n)) = ... = \mu(T^{-(n-1)}(\mathcal{A}_n))$ for all n. Let $\mu(E) = M$. Since $\mathcal{A}_n \cup T^{-1}(\mathcal{A}_n) \cup ... \cup T^{-(n-1)}(\mathcal{A}_n) = E$, we then have that $\mu(\mathcal{A}_n) = \frac{M}{n}$. Consider the set $B = \{a_z \mid |z| = 2^n \text{ for } n \in \mathbb{N}\}$. Then, there exists a subsequence of $\{T^{-i}(B)\}_{i\in\mathbb{N}}$ given by $\{T^{-i_k}(B)\}$ such that $T^{-i_k}(B)$ contains a_0 for each i_k. This shows that there exists a subsequence of the $\{T^{-i}(B)\}_{i\in\mathbb{N}}$ where the limit of the measures of the subsequence is positive.

However, consider that for any n, $B \subset \{a_z \mid z \in 2^n\mathbb{Z}\} \cup \{a_{-2^{n-1}}, a_{-2^{n-2}}, ..., a_{2^{n-1}}\}$. Note that since μ is a finite measure and E is an infinite subset of D_2, for any finite $S \subset E$, the preimages of S are disjoint and $\lim_{m\to\infty} \mu(T^{-m}(S)) = 0$. Take $S = \{a_{-2^{n-1}}, a_{-2^{n-2}}, ..., a_{2^{n-1}}\}$. Further, it is assumed that $\mu(\mathcal{A}_{2^n}) = \mu(T^{-1}(\mathcal{A}_{2^n})) = ... = \mu(T^{-(2^n-1)}(\mathcal{A}_{2^n}) = 2^{-n}M$, and so because $T^{-k}(\{a_z \mid z \in 2^n\mathbb{Z}\})$ is a subset of one of these 2^n sets,

$$\lim_{k\to\infty} \mu(T^{-k}(B)) \leq 2^{-n}M \tag{4}$$

Since we picked n arbitrarily, the limit then must be 0. The two subsequences converge to different values, giving a contradiction. □

The central part of the above argument is collapsing the structure of the chain-tree in such a way that its structure mirrors the integers. We may instead expand this to collapse a single level of the chain-tree to a set B_z corresponding to the element of the chain a_z. This reduces our considerations to the right-shift on the integers. We may leverage this even further, putting the measure into a more desirable Cesaro limit form common to Ergodic Theory.

Definition 9. *Let $(\Omega, \mathcal{F}, \nu)$ be a probability measure space and $V : \Omega \to \Omega$ a $\mathcal{F}$-measurable map. Then, we say ν is asymptotically mean stationary with respect to V if $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \nu(V^{-n}(A))$ exists for all $A \in \mathcal{F}$.*

Proposition 10. *Let there exists a finite measure μ on $\mathbb{N}$ such that $\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu(T^{-i}(A))$ exists for all $A \subset \mathbb{N}$, that is to say that $(\mathbb{N}, P(\mathbb{N}), \mu, T)$ is asymptotically mean stationary. Then, D_2 must have measure 0.*

Remark: Note that $\frac{1}{k}\sum_{i=1}^{k} \mu(T^{-i}(\mathcal{A}_n))$ converges to $\frac{1}{n}$ as $k \to \infty$. In other words, this is a much weaker requirement on the measure than the previous case. The limit does act similarly on finite sets. For any set A such that $\sum_{i=1}^{\infty} \mu(T^{-i}(A)) = l < \infty$, $\frac{1}{k}\sum_{i=1}^{k} \mu(T^{-i}(A)) \leq \frac{l}{k}$ shows that the limit of these means is 0. Since the above property holds for singletons, it holds for finite sets as well. The effort is then extending to the infinite case in the same way.

Proof. By contradiction, let μ be such a measure and $a \in D_2$ be a point so $\mu(a) > 0$. Let $H = \{a_z\}_{z\in Z}$ and E be as in the proof of proposition 1, and assume $\mu(E) = 1$ by renormalization.

We begin by redefining a set similar to the $\mathcal{A}_n$ in concept. Let $B_z = \bigcup_{i=0}^{\infty} T^{-i}(T^i(a_z))$, so that the B_z correspond to a "level" of the chain-tree E as demarcated by the a_z.

First, we construct a set B. To begin, select N such that $\sum_{i=N+1}^{\infty} \mu(B_i) + \mu(B_{-i}) < \frac{1}{20}$, so that also $\sum_{-N}^{N} \mu(B_i) \geq \frac{19}{20}$ (these values are mostly arbitrary choices, though the first must be sufficiently small for the following argument). Let $K = 2N + 1$. Let

$$B = [B_{N+1} \cup B_{N+2} \cup ... \cup B_{3N+1}] \cup [B_{7N+4} \cup B_{7N+5} \cup ... \cup B_{11N+5}] \cup ... \cup [B_{19N+10} \cup B_{19N+11} \cup ... \cup B_{27N+13}] \cup ... \tag{5}$$

Considering levels as starting from level B_{-N-1}, we skip K levels, then, at each step, we take enough nodes so that the total number of levels included divided by the number of levels since $-N-1$ is $\frac{1}{2}$, then exclude enough that this drops to $\frac{1}{3}$, and exclude K more as a buffer. Looking at blocks of K nodes, where 1 represents inclusion and 0 exclusion, this looks like the sequence

$$0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, ... \tag{6}$$

This allows us to construct two sequences based on these choices which converge in different ways.

Consider first the value $\frac{1}{2K}\sum_{i=1}^{2K}\mu(T^{-i}(B))$. Since $T^{-1}(B_i) = B_{i-1}$, and since we move $2K$, we have that the K levels in B given by $B_{N+1}, ..., B_{3N+1}$ each take the values of $B_{-N}, ..., B_N$ precisely once in the sum $\sum_{i=1}^{2K}\mu(T^{-i}(B))$. Therefore,

$$\sum_{i=1}^{2K}\mu(T^{-i}(B)) \geq \left(\frac{19}{20}\right)\cdot K \tag{7}$$

Using similar equations, we may track preimages as they pass over the central mass of $\frac{19}{20}$ or stay within the tail mass of $\frac{1}{20}$ to construct two sequences with masses bounded as

$$\frac{1}{2K}\sum_{i=1}^{2K}\mu(T^{-i}(B)) \geq \frac{19}{20}\left(\frac{K}{2K}\right) = \frac{19}{40} \qquad \frac{1}{3K}\sum_{i=1}^{3K}\mu(T^{-i}(B)) \leq \frac{1}{20} + \frac{19}{20}\left(\frac{K}{3K}\right) \tag{8}$$

$$\frac{1}{6K}\sum_{i=1}^{6K}\mu(T^{-i}(B)) \geq \frac{19}{20}\left(\frac{3K}{6K}\right) = \frac{19}{40} \qquad \frac{1}{9K}\sum_{i=1}^{9K}\mu(T^{-i}(B)) \leq \frac{1}{20} + \frac{19}{20}\left(\frac{3K}{9K}\right) \tag{9}$$

$$\frac{1}{14K}\sum_{i=1}^{14K}\mu(T^{-i}(B)) \geq \frac{19}{20}\left(\frac{7K}{14K}\right) = \frac{19}{40} \qquad \frac{1}{21K}\sum_{i=1}^{21K}\mu(T^{-i}(B)) \leq \frac{1}{20} + \frac{19}{20}\left(\frac{7K}{21K}\right) \tag{10}$$

For the pattern on the left, we consider indices such that half of the constructed sequence up to that point (considered from $-N-1$) is included in B. In $2k$, there are k left out and k included. Similarly, in $6K$, there are k out, k in, $2k$ out, $2k$ in, leaving $3k$ in and $3k$ out. The measures of these are always at least $\frac{19}{40}$ by the same computation as for the cases shown above. The pattern on the right corresponds to going far enough forward that $1/3$ of levels since $-N-1$ are included and $2/3$ excluded. The construction of B guarantees that the corresponding values are at most $\frac{1}{20}+\frac{19}{60}$. This constructs two subsequences which must have different limits and contradicts the assumption that the limit converges. □

Remark:

i) Constructing a measure with either the property in proposition 1 or proposition 2 which is only zero on points known to be in $C \cup D_1$ shows that D_2 is empty as well.

ii) This measure has weaker requirements than that posed in [1]

iii) This proposition does not require that T be nonsingular, as the following proposition does.

A slight modification of this proposition allows for an argument based on Birkhoff's Ergodic Theorem which generalizes to the class of maps posed in [3] as well as to more general measurable maps. The second version also relies on one extra proposition due to Gray and Kieffer [5]. The proof for the proposition given below may be found in U.Krengel's book [9].

Proposition 11. *The probability system* $(\Omega, \mathcal{F}, \nu, V)$*, for* V *nonsingular, is asymptotically mean stationary if and only if the averages* $\frac{1}{N}\sum_{n=1}^{N} f(V^n(x))$ *converges* ν*-almost everywhere for each bounded, measurable function* f*.*

Proof. For the reverse direction, note that $\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_A(T^n(x))$ converges for each indicator function $\mathbb{1}_A$ and $A \in \mathcal{F}$. Thus,

$$\int_\Omega \frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_A(T^n(x)) = \frac{1}{N}\sum_{n=1}^{N}\nu(V^{-n}(A)) \tag{11}$$

converges by the Dominated convergence theorem.

For the forward direction, assume that ν is a probability measure. Then, by assumption,

$$\bar{\nu}(A) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \nu(V^{-n}(A)) \tag{12}$$

defines a measure by the Vitali-Hahn-Saks Theorem ($\nu_N(A) = \frac{1}{N}\sum_{n=1}^{N} \nu(V^{-n}(A))$ for all measurable A are absolutely continuous with respect to ν and converge).

Consider the set B_f of points ω for which $\frac{1}{N}\sum_{n=1}^{N} f(V^n(x))$ converges. The set B_f is clearly V-invariant and so $\nu(B_f) = \bar{\nu}(B_f)$, where V is measure-preserving with respect to $\bar{\nu}$, so that the Birkhoff-Khinchin theorem gives $\bar{\nu}(B_f) = 1$. □

Proposition 12. *Let there exists a finite measure μ on $\mathbb{N}$ everywhere nonzero such that $(\mathbb{N}, P(\mathbb{N}), \mu, T)$ is asymptotically mean stationary. Then, $D_2 = \emptyset$.*

Remark: Note that T is nonsingular with respect to μ in this case.

Proof. The decomposition of proposition 8 shows that the set D_2 is an at-most countable union of wandering sets, or

$$D_2 = \bigcup_{n=0}^{\infty} W_n \tag{13}$$

where $\mu(T^i(W_n) \cap T^j(W_n)) = \emptyset$ for $i \neq j$.

Let m be the restriction to D_2 of the limiting measure $\bar{\mu}$ in the proof of the above proposition. Note that m is T-invariant. If $\mu(D_2) > 0$, then $m(D_2) > 0$, and in particular, $m(W_n) > 0$ for a wandering set W_n. Then, $m(\bigcup_{j=0}^{\infty} T^{-j}(W_n)) \leq m(D_2) < \infty$, but $m(\bigcup_{j=0}^{\infty} T^{-j}(W_n)) = \sum_{j=1}^{\infty} m(W_n) = \infty$, a contradiction. □

This direction of the argument, that the existence of a measure implies something about D_2, is the more important direction because it allows for statements on the Collatz map. However, the converse is also true. The following argument immediately generalizes from the Collatz map to any Syracuse-type map such that the preimage of a finite set is finite.

Lemma 13. *If D_2 is empty, there exists an everywhere-nonzero finite measure μ asymptotically mean stationary with respect to the Collatz map.*

Proof. Recall the decomposition $N = C \cup D_1 \cup D_2$ (where we assume $D_2 = \emptyset$ here). Let $C = \bigcup_{i=1}^{\infty} C_i$ where fore each i, $C_i = \{c_1, .., c_{n_i}\}$ is a cycle.

Then we construct μ to be a probability measure. Set $\mu(c_1) = ... = \mu(c_{n_i}) = \frac{1}{2^{i+2}n_i}$. Then, for $k \geq 0$, we take the $\mu(T^{-k}(T^{-1}(c_j)\backslash C_i)) = \frac{1}{2}\mu(T^{1-k}(T^{-1}(c_j)\backslash C_i))$ where all values in this set have equal measure. This is to say that we consider the branch of D_1 mapping to each c_j in this cycle (without intersecting the cycle elsewhere before mapping to c_j) and weight these equally across the c_j values. It is immediate that $\mu(\mathbb{N}) = \sum_{i=1}^{\infty} 2\mu(C_i) = 1$.

Consider first $A \subset D_1$. Then, $\sum_{j=0}^{\infty} \mu(T^{-j}(A)) < \infty$ implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N} \mu(T^{-j}(A)) = 0 \tag{14}$$

Next, assume that A is a subset of a single cycle $C_i = \{c_1, ..., c_{n_i}\}$. We have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N} \mu(T^{-j}(A \cap C_i)) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N} \mu(T^{-j}(\{c_{j_1}, ..., c_{j_l}\})) \tag{15}$$

$$= \lim_{N\to\infty} \frac{l}{N} \left(\sum_{m=0}^{N} \frac{2^{-m}(N-m)}{2^{i+2}n_i} \right) = l(\sum_{m=0}^{\infty} \frac{2^{-m}}{2^{i+2}n_i}) = \frac{l}{2^{i+1}n_i} \tag{16}$$

The argument immediately generalizes to any subset of C with a bit more algebra in computing the actual limit. Therefore, since any subset of the natural numbers may be decomposed $A = (A \cap C) \cup (A \cap D_1)$, μ is asymptotically mean stationary with repect to T. $\square$

3 The Triangle Conjecture

The measure-theoretic considerations noted above brought up the chain and the family structure, along with some of its applications. Investigating those same considerations led to the observation of some structures that organize the preimages. The strongest structure is conjectured in this section, while another notable structure is mentioned and used in a connected result.

In the case that this conjecture holds, it would partially locate the set D_2 within the chain-tree, allowing for more constructive computation on the generation of such measures or on the density of the set D_2 and the related set L of Terras. The latter will be investigated further outside of the context of the Triangle Conjecture in the next section.

3.1 The Triangle Conjecture and Extended Periodicity

The Triangle Conjecture is an a relatively easy-to-state comment on the combinatorial structure of preimages under the Collatz map, based on the family structure in Section 2.3.

To begin, consider a $\mathbb{N}_2$ node $3^k h - 1$ where we do not place the prior restrictions on h. Consider the preimages $A = \bigcup_{i=0}^{k} T^{-i}(3^k h - 1)$. This is the "triangle", since under the graphic structure shown in Figure 1, the preimage tree looks similar to a triangle.

Let $a \in A$ be such that $a \neq 3^{k-m} 2^m h - 1$ for any $m \leq k$, so that a is not on the leftmost branch of the triangle. The conjecture states that there exists $m \leq k$ so $T^k(a) < a$.

As noted above, this conjecture would completely locate the compliment of the set $L = \{x \in \mathbb{N} \,|\, T^k(x) < x \text{ for some } k \in \mathbb{N}\}$. Consider any node $x \in D_2$. Then, the set $B = \bigcup_{i=1}^{\infty} T^i(x)$ must have some minimum value $b \in L^C$, so that $L^C \cap B \neq \emptyset$. For any a in this intersection, we also have that any preimage of a would be in D_2. Thus, this conjecture would stand to locate these values as a type of indicator of chain-trees in D_2.

We now define some structures original to Terras [20], and use them to present a reduction of the proof of the triangle conjecture to a fixed h, as well as some observational evidence for the Triangle Conjecture.

Definition 14. *For $x \in \mathbb{N}$, $E_k(x) \in \{0,1\}^k$ is the vector with 0 in place n if $T^n(x)$ is even and 1 if $T^n(x)$ is odd. Further, set $S_k(x) = \sum_{i=1}^{k} (E_k(x))_i$ to be the number of 1s in the vector $E_k(x)$.*

In the same paper, Terras showed that $E_k(x) = E_k(y)$ if and only if $x \equiv y \mod 2^k$. This is precisely the periodicity of the Collatz Map. It may be restructured in a way more useful to the triangle.

Proposition 15. *The structure of the triangle generated by $3^k h - 1$ is invariant with respect to h. That is to say, for each $a \in T^{-l}(3^k h - 1)$, $1 \leq l \leq k$, and for any $h_1 \in \mathbb{N}$, there exists $a_1 \in T^{-l}(3^k h_1 - 1)$ such that $E_l(a) = E_l(a_1)$.*

Proof. Fix $k, h \in \mathbb{N}$ and consider $a \in T^{-l}(3^k h - 1)$ for $1 \leq l \leq k$. We wish to show that there exist α, β not dependent on h so that $a = 2^l 3^{k-l} h\alpha + \beta$. This decomposes a into the part maintaining the initial power of 3, $2^l 3^{k-l}\alpha$, and the "remainder" part that helps locate it on the branch β. Indeed, consider that the preimage of $3^k h - 1$ is $\{2(3^{k-1})h - 1, 2(3^k h) - 2\}$, and this is true for the first preimage. Working inductively, if we assume there are α_0 and β_0 not dependent on h so $T(a) = 2^{l-1} 3^{k-l+1} h\alpha_0 + \beta_0$, then $a \in \{2^l 3^{k-l} h\alpha_0 + \frac{2\beta_0 - 1}{3}, 2^l 3^{k-l} h(3\alpha_0) + 2\beta_0\}$ where, when the first option is possible, $\frac{2\beta_0 - 1}{3}$ is an integer. Therefore, we may express a in the desired form as well.

Now, consider $E_l(a)$. Denote, for arbitrary $h_1 \in \mathbb{N}$, $a_1 = 2^l 3^{k-l} h_1 \alpha + \beta$, such that $a_1 \equiv a \mod 2^l$. By Terras' periodicity result, $E_l(a_1) = E_l(a)$, and further this implies by repeated applications of the Collatz map that $T^l(a_1) = 3^k h_1 - 1$.

□

The above structure result is to say that, with the triangle displayed as in Figure 1, changing h amounts to changing the values but not the graph associated to the triangle. This allows us to look at one specific case:

Proposition 16. *For a fixed k, the value $a \in T^{-k}(3^k h-1)$ has an associated node a_1 so $T^m(a_1) > a_1$ for all $1 \leq m \leq k$ if and only if*

$$3^{S_m(a)} > 2^m \tag{17}$$

for the same m

Proof. In the case $k = 1$, $\frac{T(a)}{a}$ is either $1/2$ or $\frac{3}{2} - \frac{1}{2(2^k h\alpha+\beta)} < \frac{3}{2}$. Therefore, in the case that $a < T^m(a)$

$$\frac{T^m(a)}{a} = \frac{T^m(a)}{T^{m-1}(a)} \frac{T^{m-1}(a)}{T^{m-2}(a)} \cdots \frac{T^2(a)}{T(a)} \frac{T(a)}{a} < (\frac{1}{2})^{m-S_m(a)} (\frac{3}{2})^{S_m(a)} = \frac{3^{S_m(a)}}{2^m} \tag{18}$$

Consider now that $3^{S_m(a)} > 2^m$ for all such $m \leq k$. Then, we may take some $\alpha = \min\{a, T(a), ..., T^k(a)\}$ and the same computation above gives

$$\frac{T^m(a)}{a} > 2^{-m} \left(3 - \frac{1}{\alpha}\right)^{S_m(a)} \tag{19}$$

Since $\alpha \to \infty$ as $h \to \infty$, for sufficiently large h, we then have that the right is greater than 1 by the assumption. Denote such an h as h_1 and the associated point on the tree (as in the previous proposition) as a_1, finishing the proof. □

Thus, we may reduce to consider the structure in a single case, say $h = 1$, and looking at the corresponding limiting ratio $3^{S_k(a)}/2^k$. For $k \leq 100$, this has shown no counterexamples to the triangle conjecture as checked by direct computation. Above this point, the computational effort to compute the tree begins to become burdensome for a standard computer.

3.1.1 Extending Periodicity for Syracuse Maps

In the case of a Syracuse map, we may extend the periodicity and structure results directly. This simply allows for more play when it comes to density results later.

We may redefine the vector $E_k(x)$ to live in the space $\{0, 1, 2, ..., d-1\}^k$ so $E_k(x)_i = j$ if and only if $V^{i-1}(x) \equiv j \mod d$.

Proposition 17. $E_k(x) = E_k(y)$ *if and only if* $x \equiv y \mod d^k$

Proof. If $x \equiv y \mod d^k$, $x = y + md^k$ for some integer m, so that $a \equiv y \mod d$ and $V(x) = \frac{m_{i_1}x+r_{i_1}}{d} = \frac{m_{i_1}(y+md^k)+r_{i_1}}{d} = \frac{m_{i_1}y+r_{i_1}}{d} + m'd^{k-1}$ for $m' = m_i m$. Hence, $V(x) \equiv V(y) \mod d^{k-1}$ and we may step iteratively across the vector to show $E_k(x) = E_k(y)$.

Furthermore, if $E_k(x) = E_k(y)$, we have $x \equiv y \mod d$ immediately. Let us pick $l_0, l_1, ..., l_k$ which are not divisible by d so $x = y + l_0 + l_1 d + l_2 d^2 + ... + l_k d^k$. The previous step implies that $l_0 = 0$. Since $E_k(x)_1 = E_k(y)_1$ and

$$\frac{m_{i_1}x + r_{i_1}}{d} = \frac{m_{i_1}(y + l_1 d + l_2 d^2 + ... + l_k d^k) + r_{i_1}}{d} = \frac{m_{i_1}y + r_{i_1}}{d} + m_{i_1}(l_1 + l_2 d + ... + l_k d^{k-1})$$

as well as the fact that m_{i_1} is relatively prime to d, we have that $l_1 = 0$. Repeating this iteratively as well shows that $x \equiv y \mod d^k$.

□

This extends immediately to the triangle structure as well, by the same argument as for the Collatz map. This is not as important to the triangle, but will be used for density results later.

3.2 General Results on the Tree

The study of the triangle and its relative density relies on the size of the triangle. In this section, we showcase some results on how the size of the preimage triangle $\bigcup_{i=1}^{k} T^{-i}(3^k h - 1)$ grows. The main result of this section that this growth is eventually geometric. To be precise, let A_n be the cardinality $|\bigcup_{i=1}^{n} T^{-i}(3^k h - 1)|$ for any n, $k > 0$ and $h \geq 1$. Then, we show

Proposition 18. *There exist sequences R_n and B_n such that $R_n \leq A_n \leq B_n$ and*

$$R = \lim_{n\to\infty} \frac{R_{n+1}}{R_n} \text{ and } B = \lim_{n\to\infty} \frac{B_{n+1}}{B_n} \tag{20}$$

exist and $1 < R < B < 2$.

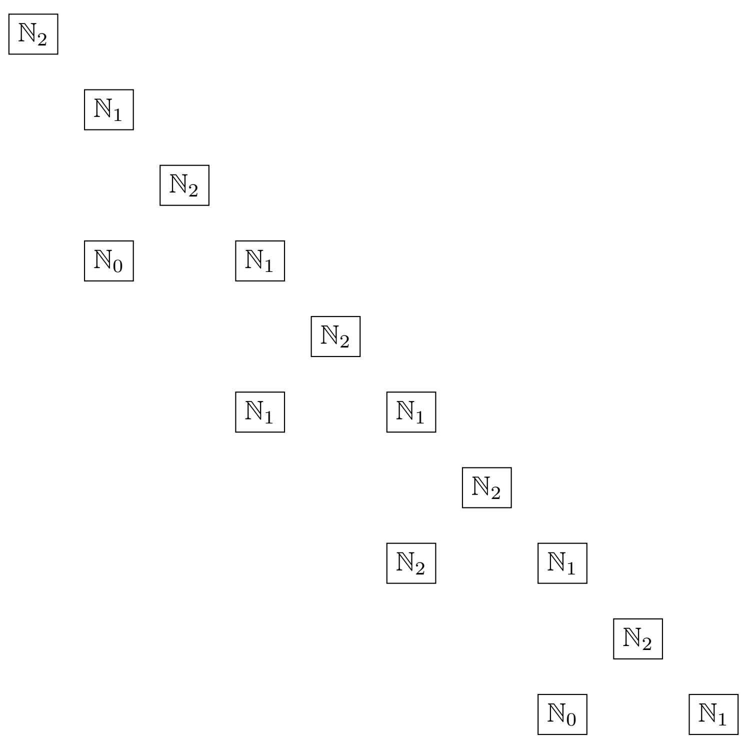
We may actually go as far as producing values for these limits. This proposition will be proved through 3 steps, first by noting some structural properties of the tree, second by turning this into lower and upper bounds on the size of the level A_n, and third by showing that these limits truly exist for the bounds given.

3.2.1 Some Preimage Structures

Let us begin by considering a $\mathbb{N}_2$ node. A $\mathbb{N}_0$ node has an uninteresting preimage since $T^{-1}(a) = 2a$ for any $\mathbb{N}_0$ node, where $2a$ is again $\mathbb{N}_0$, and any $\mathbb{N}_1$ node is the image of a $\mathbb{N}_2$ node, so this is sufficient to describe all cases.

Recall that we refer to the preimages $T^{-1}(a) = \{\frac{2a-1}{3}, 2a\}$ as the left and right preimages, and the right branch refers to iteratively taking right preimages. In the case we begin with a $\mathbb{N}_2$ node, this produces a sequence of nodes $\mathbb{N}_2, \mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_1, ...$ In counting these, we will see that each $\mathbb{N}_2$ node will also produce a left preimage in the next preimage of a, but each $\mathbb{N}_0$ or $\mathbb{N}_1$ node only produces a right preimage.

In fact, we may refine this further. The $\mathbb{N}_2$ nodes appearing via the right-most branch, the left preimage of those repeats in a sequence $\mathbb{N}_0$, $\mathbb{N}_1$, $\mathbb{N}_2$. See the diagram below.



The proof of this pattern is also relatively quick. For a seed node $3^k h - 1$, the $\mathbb{N}_2$ nodes on the rightmost branch are of the form

$$2^{2l}(3^k h - 1) = 3(3^{k-1}2^{2l}h - \frac{2^{2l}-1}{3}) - 1 \tag{21}$$

so that the left node is of the form

$$3^{k-1}2^{2l+1}h - \frac{2^{2l+1}-2}{3} - 1 \tag{22}$$

we assume $k > 2$ since the other cases are easy to compute, so the remainder term modulo 3 is $-\frac{2^{2l+1}-2}{3} - 1$. By considering that the powers of 2 modulo 9 are, in order $2, 4, 8, 7, 5, 1$, we get that when $2l \equiv 2 \mod 6$, $-\frac{2^{2l+1}-2}{3} - 1 \equiv 0 \mod 3$, when $2l \equiv 4 \mod 6$, $-\frac{2^{2l+1}-2}{3} - 1 \equiv 1 \mod 3$, and when $2l \equiv 0 \mod 6$, $-\frac{2^{2l+1}-2}{3} - 1 \equiv 2 \mod 3$.

3.2.2 Size Bounds

Lemma 19. *We may bound the cardinality of $T^{-m}(3^k h - 1)$ from below by a number $R(m) + 1$ so that $R(0) = 0$, $R(1) = 1$, and $R(m)$ satisfies*

$$R(m) = 1 + \lfloor \frac{k-1}{2} \rfloor + \sum_{l=0}^{\lfloor \frac{m-6}{6} \rfloor} R(m - 6 - 6l) + \sum_{j=0}^{\lfloor \frac{m-7}{6} \rfloor} R(m - 7 - 6j) \tag{23}$$

Proof. We begin by taking a central assumption: we assume that the left-branch of a $\mathbb{N}_2$ node always produces a $\mathbb{N}_0$ node. In the real case, it may produce a $\mathbb{N}_1$ or $\mathbb{N}_2$ node as well, but these always produce more than 1 preimage, so that a $\mathbb{N}_0$ node undercounts the number of preimages and gives a lower bound.

We incorporate the above result and then track more complexity on the right hand side. Let $R(m)$ denote the number of nodes produced by the right-hand branch under these assumptions, so $R(0) = 0$ and $R(1) = 1$ through direct analysis of the pattern.

Using the structure of the previous lemma, we may generate the recurrence relation given by tracking the the rightmost branch (the value 1), the number of times we get some addition to the left ($\lfloor \frac{k-1}{2} \rfloor$), the number of times an $\mathbb{N}_1$ node of the above pattern contributes extra right-hand branch nodes ($\sum_{l=0}^{\lfloor \frac{m-6}{6} \rfloor} R(m - r - 6l)$, since it begins the level *after* the $\mathbb{N}_1$ node), and the number of times the $\mathbb{N}_2$ nodes contribute similarly ($\sum_{j=0}^{\lfloor \frac{m-7}{6} \rfloor} R(m - 7 - 6j)$).

□

Parallel to the above case, we may compute an upper bound by taking a central assumption that any $\mathbb{N}_2$ node produces another $\mathbb{N}_2$ node on its left branch, overcounting the number of nodes produced.

Lemma 20. *We may bound the cardinality of $T^{-m}(3^k h - 1)$ from above by*

$$B(m) = F(m+1) + F(m) = \lfloor \frac{\phi^{m+1}}{\sqrt{5}} + \frac{1}{2} \rfloor + \lfloor \frac{\phi^m}{\sqrt{5}} + \frac{1}{2} \rfloor \tag{24}$$

where $F(m)$ denotes the m-th Fibonacci number and ϕ the golden ratio.

Proof. This comes from the fact that each $\mathbb{N}_1$ node becomes an $\mathbb{N}_2$ node at the subsequent level, and each $\mathbb{N}_2$ adds both a $\mathbb{N}_1$ and $\mathbb{N}_2$ node under these assumptions. Therefore, the number of N_1 nodes is the number of N_2 nodes in the previous level, and so the number of N_2 nodes at a given level, denoted $N(m)$ is precisely $N(m-1) + N(m-2)$, the defining relation for the Fibonacci sequence. However, starting with 1. We simply readjust indexing because the Fibonacci sequence starts with its 0th term being 0 but we start with it being 1. □

3.2.3 Nearly-Geometric Growth of the Tree

Lemma 21. *For $R_n = R(n)$ defined as in lemma 19*

$$\lim_{n\to\infty} \frac{R_{n+1}}{R_n} \tag{25}$$

exists and is greater than 1.

Proof. Let us first consider the minimal growth case. From the relation

$$R(m) = 1 + \lfloor \frac{k-1}{2} \rfloor + \sum_{l=0}^{\lfloor \frac{m-6}{6} \rfloor} R(m-6-6l) + \sum_{j=0}^{\lfloor \frac{m-7}{6} \rfloor} R(m-7-6j) \tag{26}$$

we may obtain the easier recurrence relation $R(m+6) - R(m) = 3 + R(m) + R(m-1)$ or $R(m+6) = 3 + 2R(m) + R(m-1)$. We then need only compute $R(0) = R(1) = R(2) = 1$ and $R(3) = R(4) = 2$, $R(5) = R(6) = 3$ to compute the entire sequence. We now denote $R_n = R(n)$.

Taking motivation from the Fibonacci numbers as in [16], we may associate a polynomial to the recurrence relation and look at its roots to find this limit.

In particular, note that we may rewrite $R(m+6) = 3+2R(m)+R(m-1)$ to be $R(m+8)-R(m+7) = 2R(m+2)-R(m+1)-R(m)$ giving $R(m+8)-R(m+7)-2R(m+2)+R(m+1)+R(m) = 0$. According to the Bernoulli trick of the above reference, we may then represent $R(m) = \sum_{i=1}^{8} a_i(c_i)^m$ for c_i the roots of the associated polynomial $x^8 - x^7 - 2x^2 + x + 1$ if this polynomial is separable.

We may note in particular that this polynomial is $(x-1)(x^7 - 2x - 1)$. Note that 1 is not a root of $x^7 - 2x - 1$. Further, $x^7 - 2x - 1$ is separable if and only if it shares no roots with its derivative, $7x^6 - 2$. However, the roots of the derivative are of the form $\sqrt[6]{2/7}e^{\frac{2\pi ik}{6}}$ for $k = 1, 2, 3, 4, 5, 6$. Then,

$$(\sqrt[6]{2/7}e^{\frac{2\pi ik}{6}})^7 - 2(\sqrt[6]{2/7}e^{\frac{2\pi ik}{6}}) - 1 = \tag{27}$$

$$\sqrt[6]{2/7}e^{\frac{2\pi ik}{6}}(\frac{2}{7}) - 2(\sqrt[6]{2/7}e^{\frac{2\pi ik}{6}}) - 1 = \tag{28}$$

$$\sqrt[6]{2/7}e^{\frac{2\pi ik}{6}}(\frac{12}{7}) - 1 \tag{29}$$

This cannot be 0 because of the differing magnitudes of the two values, so that the polynomial is indeed separable.

Note secondarily that the polynomial $x^7 - 2x - 1$ has a root of largest magnitude which is real, and approximately 1.19. This may be located by Rouche's theorem or Newton's method.

This representation $R(m) = \sum_{i=1}^{8} a_i(c_i)^m$ shows that the ratio $\frac{R_{m+1}}{R_m}$ then approaches the value of this largest root. In particular, this limit exists and is greater than 1. ∎

In the case of the upper-bound, the fact that we relied on Fibonacci numbers gives an immediate limiting ratio of $\phi \approx 1.618$. The pair of these limits then proves Proposition 18.

3.2.4 Relation to the Triangle as a Whole

The asymptotically geometric growth of the lower and upper bounds has a secondary goal: it shows that the "lowest level" of the triangle, i.e. $T^{-n}(3^k h - 1)$ in the triangle $\bigcup_{i=1}^{n} T^{-i}(3^k h - 1)$, takes up a consistent portion of the triangle as $n \to \infty$. This follows directly from a lemma:

Lemma 22. *Let a_n be any sequence such that $\frac{a_{n+1}}{a_n} \to \alpha > 1$ as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \frac{a_n}{\sum_{i=1}^{n} a_i} \tag{30}$$

converges to a nonzero value. In particular, it converges to $\frac{\alpha}{\alpha-1}$.

Proof. Pick some bound $\epsilon < (\alpha - 1)$. Then, we may find N such that for $n \geq N$, $|\frac{a_{n+1}}{a_n} - \beta| < \epsilon$. For $k > N$ we have that

$$\frac{a_k}{\sum_{i=1}^{k} a_i} = \frac{1}{\frac{\sum_{i=1}^{N-1} a_i}{a_k} + \sum_{i=N}^{k} \frac{a_i}{a_k}} \tag{31}$$

and thus that

$$\frac{1}{(K/a_k) + \sum_{i=0}^{N-k} (\frac{1}{\beta+\epsilon})^j)} \leq \frac{a_k}{\sum_{i=1}^{k} a_i} \leq \frac{1}{(K/a_k) + \sum_{i=0}^{N-k} (\frac{1}{\beta-\epsilon})^j)} \tag{32}$$

since $\beta > 1$, we have that $K/a_k \to 0$ as $k \to \infty$ and so

$$\frac{\beta + \epsilon}{\beta + \epsilon - 1} \leq \liminf \frac{a_k}{\sum_{i=1}^{k} a_i} \leq \limsup \frac{a_k}{\sum_{i=1}^{k} a_i} \leq \frac{\beta - \epsilon}{\beta - \epsilon - 1} \tag{33}$$

Again, since $\beta > 1$, we may let $\epsilon \to 0$ to obtain the result. □

4 Density Results

Section 1 Introduced a strict viewpoint for proving the Collatz Conjecture, which is a difficut endeavor despite the simplification. Section 2 begins to look at how the involved structures might tie to the density of D_2 by looking at how nodes in the complement of Terras' set are placed within the tree and their rate of occurrence. We now shift strictly over to density, inspired by trying to find the number of candidates for failure in the triangle conjecture, with the motivation of bounding these strictly for more density arguments in that regard. However, the involved method extends to prove some other, important results. The following work was presented originally in the author's work [2].

4.1 A Rate of Convergence of the density of the set L

Recall that from [20], we pull the set

$$L = \{x \in \mathbb{N} \mid T^m(x) < x \ \text{ for some } \ m \in \mathbb{N}\} \tag{34}$$

This set may be further refined to

$$L_k = \{x \in \mathbb{N} \mid T^m(x) < x \ \text{ for some } \ m \leq k\} \tag{35}$$

Further, recall the E_k vectors:

Definition 23. *For $k, y \in \mathbb{N}$, define $E_k(y)$ to be the vector of length k whose i^{th} component is 1 if $T^{i-1}(y)$ is odd, and 0 if it is even. Define $S_k(y)$ to be the sum of the elements of $E_k(y)$.*

With these structures, we now prove the following refinement of Terras' density theorem.

Theorem 24. *For fixed $k \in \mathbb{N}$, let $L_k = \{y \in \mathbb{N} \,|\, \exists\, m,\ 1 \leq m \leq k,$ such that $T^m(y) \leq y\}$. The density of L_k^C is at most*

$$\frac{2^m}{2^k} \prod_{n=0}^{m} \frac{2n+1}{n+1} \tag{36}$$

where $m = \lfloor \frac{k}{2} \rfloor$.

To prove this theorem, we will take 3 steps. First, we will introduce a general structure used in this section and the next, which has the form of a Pascal or Catalan triangle with a set of restrictions. Second, we will connect this general form to a specific triangle related to the Collatz map. Third, we will use the structure of the triangle to compute the upper bound.

Second, we will look at a specific case of this triangle and some basic results about it. Third, we will connect this to the Collatz map.

Step 1: Take a map $\tau : \mathbb{N} \to \mathbb{R}$. We define a sequence of sequences, $\{\{x_i^n\}_{i \geq 0}\}_{n \geq 0}$ where $x_0^n = 1$ for all n and for $n > 1$,

$$x_k^n = \begin{cases} x_k^n + x_{k-1}^n & k \leq \tau(n) \\ \\ 0 & \text{else} \end{cases} \tag{37}$$

Consider the n^{th} sequence to correspond to the n^{th} row of the constructed triangle. For example, if we take $\tau(n) = n$, then this defines the standard Pascal Triangle. The function τ restricts when the rows may expand to have more nonzero values in the sequence.

Consider, for example, the first 11 rows of the triangle constructed by $\tau(n) = \frac{n}{2}$ (starting at the 0^{th} row).

	$i =$	0	1	2	3	4	5...	
$n = 0$		1	0	0	0...			
$n = 1$		1	0	0	0...			
$n = 2$		1	1	0	0...			
$n = 3$		1	2	0	0...			
$n = 4$		1	3	2	0	0...		
$n = 5$		1	4	5	0	0...		
$n = 6$		1	5	9	5	0	0...	
$n = 7$		1	6	14	14	0	0...	
$n = 8$		1	7	20	28	14	0	0...
$n = 9$		1	8	27	48	42	0	0...
$n = 10$		1	9	35	75	90	42	0	0...
		$\vdots$						

Step 2: We prove the following lemma

Lemma 25. *Let $\tau(k) = 2k$. Then, the row sum $\sum_{i=0}^{\infty} x_i^n$ gives an upper bound for the number of vectors $S_k(x)$ such that $T^m(x) > x$ for all $1 \leq m \leq k$.*

Proof. Consider $y \in L_k^C$. Then, we have that $\frac{T^n(y)}{y} > 1$ for all $1 \leq n \leq k$. Set $l = S_n(y)$ and and in particular that for all such n

$$(\frac{3y+1}{2y})^l(\frac{y}{2y})^{n-l} \geq \frac{T^n(y)}{T^{n-1}(y)}\frac{T^{n-1}(y)}{T^{k-2}(y)}...\frac{T(y)}{y} = \frac{T^n(y)}{y} \geq 1 \tag{38}$$

By taking logs, we note that

$$S_n(y) \geq \frac{n\ln(2)}{\ln(3+\frac{1}{y})} \tag{39}$$

so that $S_n(y) \geq \frac{n}{2}$ in general since $y \geq 1$. Therefore, the number of vectors satisfying this inequality for $n \leq k$ bounds the number of vectors so $T^n(y) > y$ for $n \leq k$.

To count the number of $E_k(x)$ vectors satisfying this restriction, we may construct a recurrence relation by counting the number of 0s possible in the vector. There is always only a single vector with no 0s, the vector of all 1s. If the number of 0s, l, is less than $\frac{n}{2}$, then when considering a E_{n+1} vector, we may take a vector with l zeroes and attach another 0 or add a 1 at the end while still satisfying this inequality. Therefore, the number of E_{n+1} vectors with l zeroes, $1 \leq l \leq \frac{n}{2}$, is the number of E_n vectors with l zeroes plus the number with $l-1$ zeroes. Now, consider that if $l > \frac{n}{2}$ and if $l \leq \frac{n+1}{2}$, the number of E_{n+1} vectors with l zeroes is precisely the number of E_n vectors with $l-1$. For all other values, there are 0 vectors satisfying the relations. Tracing these recurrence relations shows that the number of E_n vectors with i zeroes is precisely x_i^n for $\tau(k) = 2k$. Therefore, counting these vectors reduces to taking the row sum of the triangle constructed by this τ. Fix the $\{\{x_i^n\}_{i\geq 0}\}_{n\geq 0}$ as those generated this way.

□

Notice that that row sums are strictly increasing, so it suffices to consider only the odd-number rows to generate an upper bound. From the triangle above, this amounts to considering the rows

$$
\begin{array}{llllll}
n=1 & 1 & & & & \\
n=3 & 1 & 2 & & & \\
n=5 & 1 & 4 & 5 & & \\
n=7 & 1 & 6 & 14 & 14 & \\
n=9 & 1 & 8 & 27 & 48 & 42 \\
 & \vdots & & & &
\end{array}
$$

Consider now this triangle within its own right. Define, now, a sequence of sequences for this triangle $\{\{y_i^n\}_{i\geq 0}\}_{n\geq 0}$ where $y_i^n = x_i^{2n+1}$. Considering the recurrence relation row-wise, $y_k^n = y_{k-2}^{n-2} + 2y_{k-1}^{n-2} + y_k^{n-2}$.

This triangle develops a more Catalan-like relation, and this is not the first time it has appeared. In particular, Shapiro [17] showed that for the largest index k so that $y_k^n \neq 0$, y_k^n is precisely the n^{th} Catalan number, $\frac{1}{n+1}\binom{2n}{n}$. He also computed the row sums of this triangle, which we repeat in a simplified way for posterity.

Lemma 26. *The row sum of the triangle on x_i^n at level $2n+1$ is*

$$\sum_i x_i^{2n+1} = \sum_i y_i^n = 2^n \prod_{k=0}^{n} \frac{2k+1}{k+1} \tag{40}$$

Proof. We apply induction to the triangle $\{\{y_k^n\}\}$, noting that the n-th level of this triangle corresponds to the $2n+1$-st level of the $\{\{x_k^n\}\}$ triangle.

The formula given is immediate in the case $n=0$ or $n=1$ from the values computed above. Let it be shown for values up to $n-1$ and consider row n. Then, also note that

$$\sum_i y_i^n = \sum_i y_{i-2}^{n-1} + 2y_{i-1}^{n-1} + y_i^{n-1} = 4(\sum_i y_i^{n-1}) - y_{\frac{n-1}{2}}^{n-1} = 4(\sum_i y_i^{n-1}) - y_k^{n-1} \tag{41}$$

where $\frac{n-1}{2} = k$ is the largest index i so $y_i^{n-1} = x_i^{2n-1}$ is nonzero. Now, we may apply the inductive assumption and the result from [17] to note that this sum is

$$4(2^{n-1}\prod_{k=0}^{n-1}\frac{2k+1}{k+1}) - \frac{1}{n+1}\binom{2n}{n} \tag{42}$$

With some algebraic manipulation,

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\frac{(2n)!}{n!n!}\right) = \frac{1}{n+1}\left((\frac{2}{1}\times\frac{4}{2}\times\ldots\times\frac{2n}{n})(\frac{1}{1}\times\frac{3}{2}\times\ldots\times\frac{2n-1}{n})\right) \tag{43}$$

$$= \frac{2}{n+1}\left(2^{n-1}\prod_{k=0}^{n-1}\frac{2k+1}{k+1}\right) \tag{44}$$

□

Step 3: We now prove the theorem.

Proof. The number of unique $E_k(x)$ vectors is 2^k by Terras' periodicity. The number of such vectors satisfying $S_n(x) > \frac{x}{2}$ for all $n \leq k$ is then bounded above by

$$2^m \prod_{n=0}^{m} \frac{2n+1}{n+1} \tag{45}$$

for $m = \lfloor \frac{k}{2} \rfloor$ by lemma 26 and lemma 25. Thus, the density of L_k^C is then at most

$$2^{m-k} \prod_{n=0}^{m} \frac{2n+1}{n+1} \tag{46}$$

□

The following corollary shows exactly that this does refine Terras' density theorem

Corollary 27. *The density of L is* 1.

Proof. Since $L^C \subset L_k^C$ for all k, we have that the density of L^C is at most

$$\lim_{k\to\infty} \frac{2^m}{2^k} \prod_{n=0}^{m} \frac{2n+1}{n+1} = \lim_{k\to\infty} \frac{2^m}{2^{k-m-1}} \prod_{n=0}^{m} \frac{2n+1}{2n+2} \leq \lim_{m\to\infty} 2(\prod_{n=0}^{m} \frac{2n+1}{2n+2}) \tag{47}$$

for m a function of k as above. Note then that the rightmost limit is then

$$\exp(\lim_{m\to\infty} \ln(2) + \sum_{n=0}^{m} \ln(1 - \frac{1}{2n+2}) \tag{48}$$

for $x < 1$, we have that $ln(1-x) \leq -x$ since $f(x) = \ln(1-x)+x$ has $f(0) = 0$ and $f'(x) = \frac{-x}{1-x} \leq 0$. Thus, since the exponential is an increasing function, this limit is at most

$$\exp(\lim_{m\to\infty} \ln(2) + \sum_{n=0}^{m} -\frac{1}{2n+2}) \tag{49}$$

The series is harmonic and thus diverges to negative infinity, so that the total limit is 0. □

4.1.1 Connection to the Triangle Conjecture

The above rate of density exactly locates those nodes such that $T^m(x) > x$ for all $1 \leq m \leq k$ at the general level. Since the triangle considers a specific set of the E_k vectors, it immediately implies the following connection to the triangle conjecture.

Corollary 28. *For a fixed $k, h \in \mathbb{N}$, the number of $a \in T^{-k}(3^k h - 1)$ such that for all m so $1 \leq m \leq k$ has $T^m(a) > a$ is at most*

$$2^m \prod_{n=0}^{m} \frac{2n+1}{n+1} \tag{50}$$

where $m = \lfloor \frac{k}{2} \rfloor$.

Proof. The number of nodes a in the triangle generated by $3^k h-1$ such that $T^m(a) > a$ for $1 \leq m \leq k$ is at most the number of $E_k(y)$ vectors corresponding to $1 \leq y \leq 2^k$ so $T^m(y) > y$ for $1 \leq m \leq k$. Thus, this is given precisely as in lemma 25 and lemma 26. □

5 Conclusion

The main result of Section 4 connects the combinatorial difficulty of the Collatz map to a difficult but well-studied structure. This argument can be extended to many broader triangles, allowing for use with more maps and even with the case $M_c = \{x \in \mathbb{N} \,|\, T^k(x) < x^c \text{ for some } k \in \mathbb{N}\}$ of Korec. However, it relies on an approximation of equation 38. This approximation can be sharpened by considering closer and closer rational cases such as $S_n(y) \geq n\frac{3}{5}$, with the goal considering the strictly irrational case $S_n(y) \geq n\frac{\ln(2)}{\ln(3)}$. This connects to less-well-studied structures, because of the irrationality and the tendency of combinatorics to focus on rational cases.

The open conjecture of Section 3 could indicate some set relative to each chain-tree to replace L in the case of the Collatz map, as it indicates precisely which nodes would be poorly behaved with respect to L. These considerations within specific trajectories also lend to further analysis of the Collatz map as a dynamical system, where stronger methods such as those in Section 2 may be involved. Indeed, the Triangle Conjecture or its derivative density results could provide an avenue to construct a measure to satisfy the Cesaro-limit case or weaker requirements to show the trajectories are bounded.

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